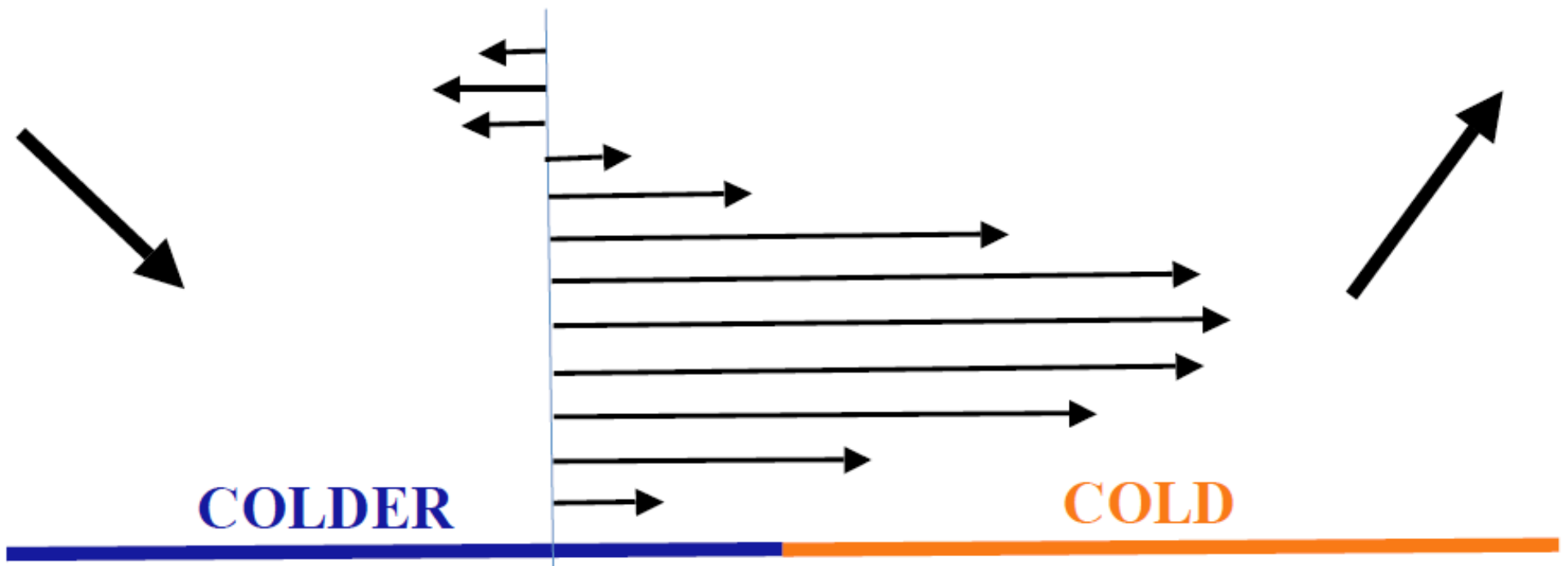


Horizontal buoyancy-driven flow along a differentially cooled underlying surface

By Alan Shapiro and Evgeni Fedorovich

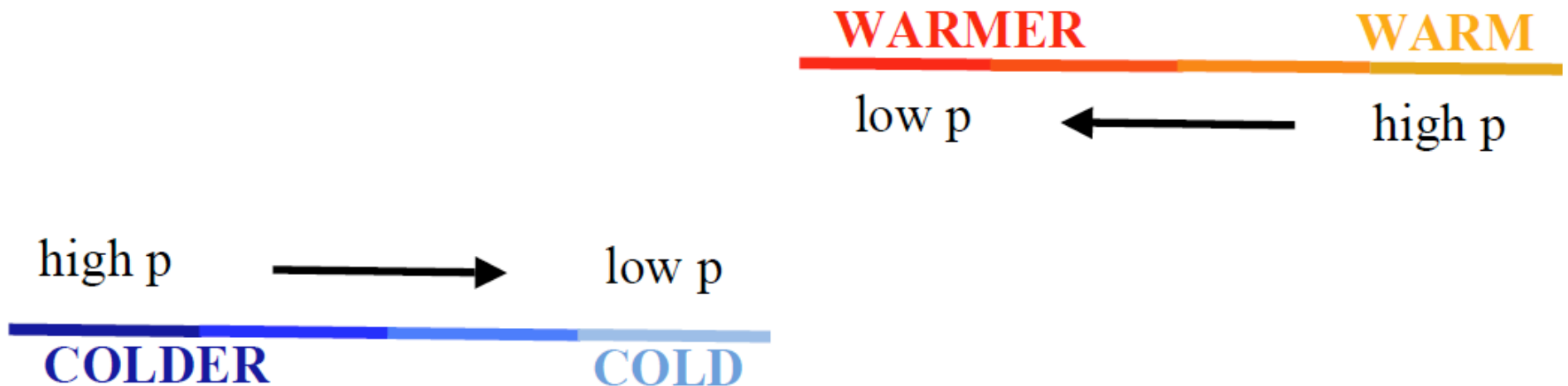
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6th Baltic Heat Transfer Conference, 24–26 August 2011, Tampere, Finland.



Horizontal natural convection

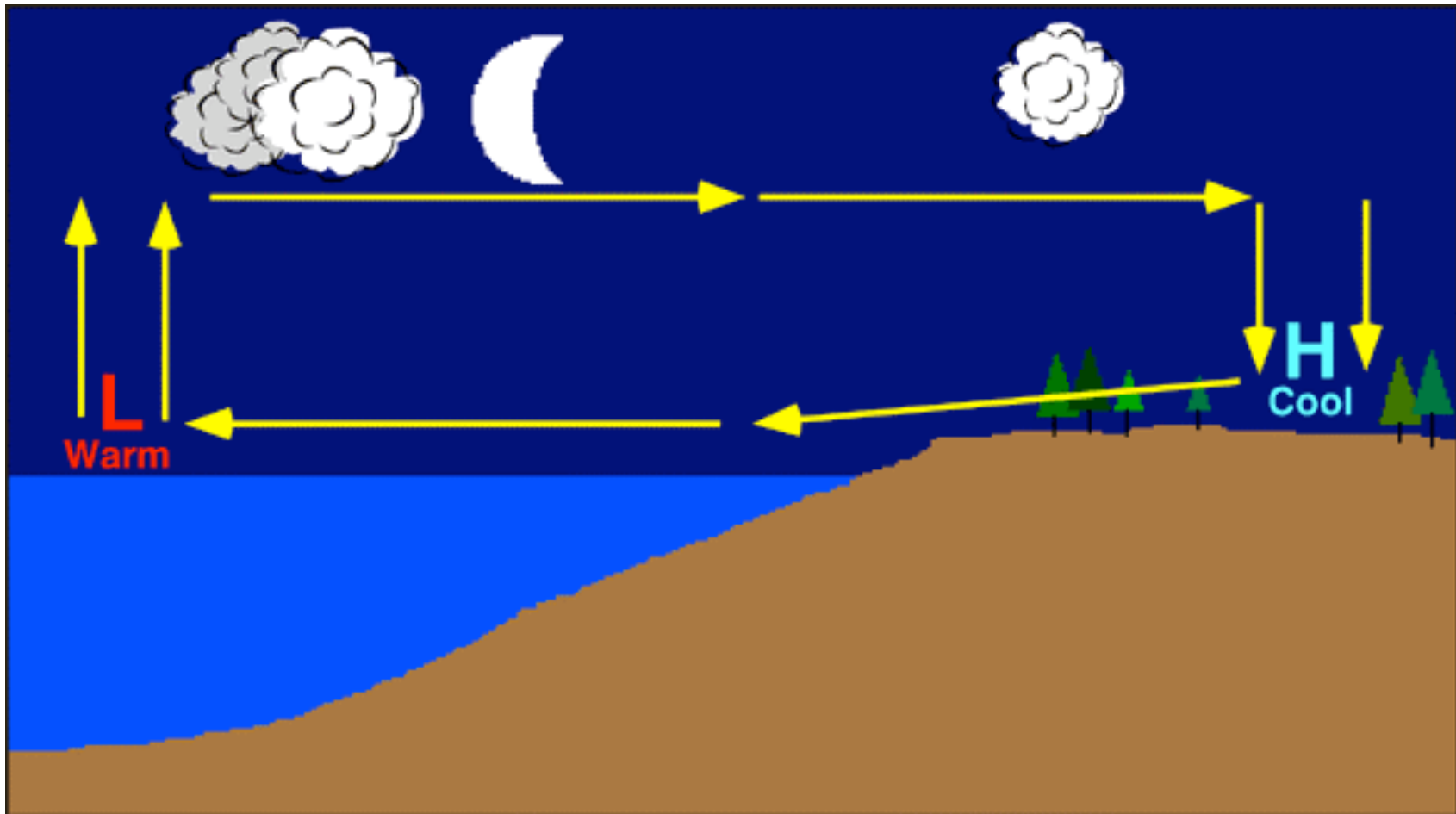
Suppose an underlying surface is differentially cooled or, equivalently, an overlying surface is differentially warmed.



In either case, **buoyancy acts in a direction perpendicular to the expected principal motion**. Buoyancy drives the flow only indirectly – through the pressure field, which responds hydrostatically to buoyancy.

Such situations arise in engineering problems (e.g., cooling of electronic circuitry) and in meteorology (e.g., land and sea breezes, density currents).

The land breeze

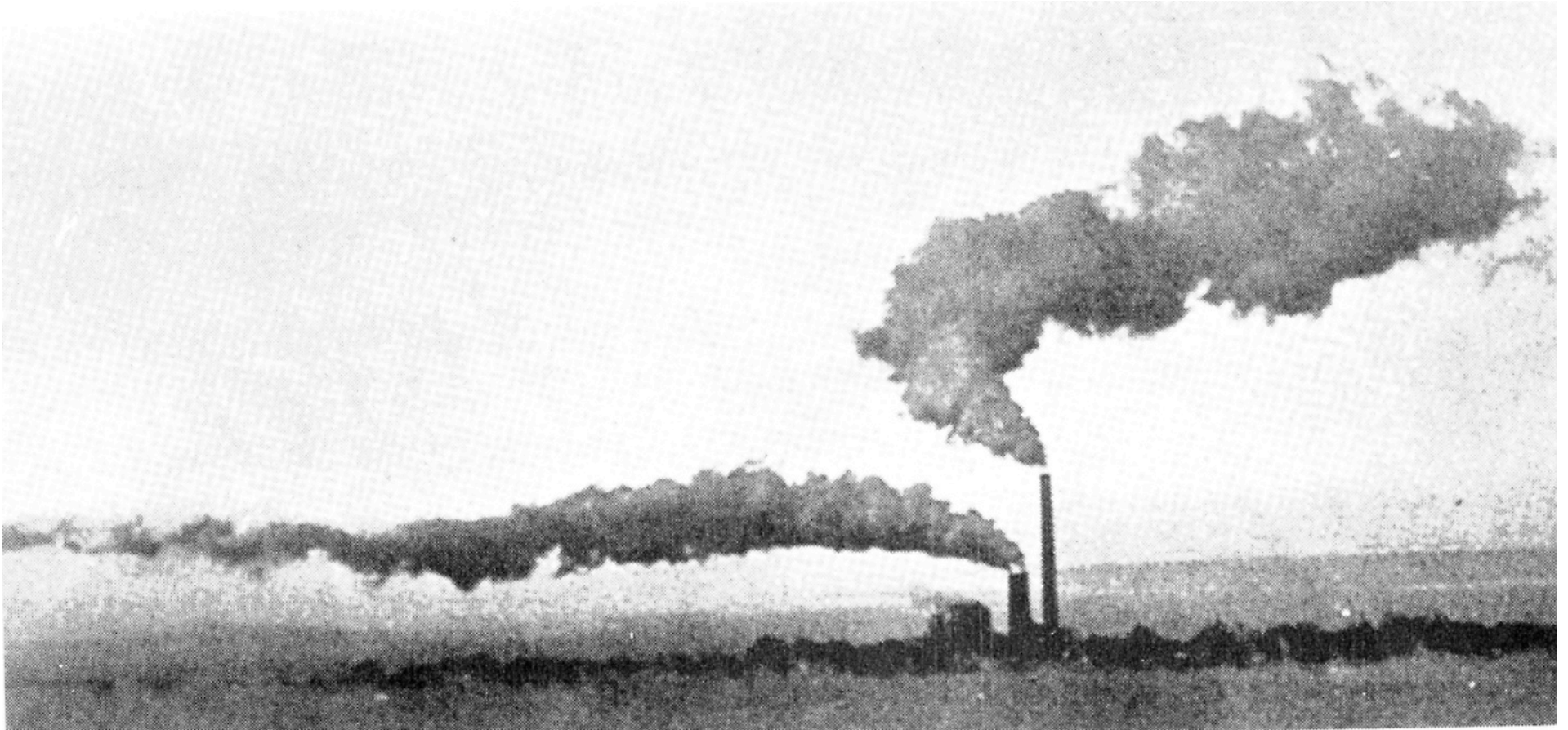


[From Pidwirny, M. (2006) "Local and Regional Wind Systems"]

After sunset, the land and sea begin to cool. But the land cools faster than the sea, and a lateral temperature contrast develops. The hydrostatic pressure field associated with this contrast drives a flow toward the sea.

Sea breeze

The opposite situation develops after sunrise and leads to a sea breeze.



[Photo by Ralph Turncote]

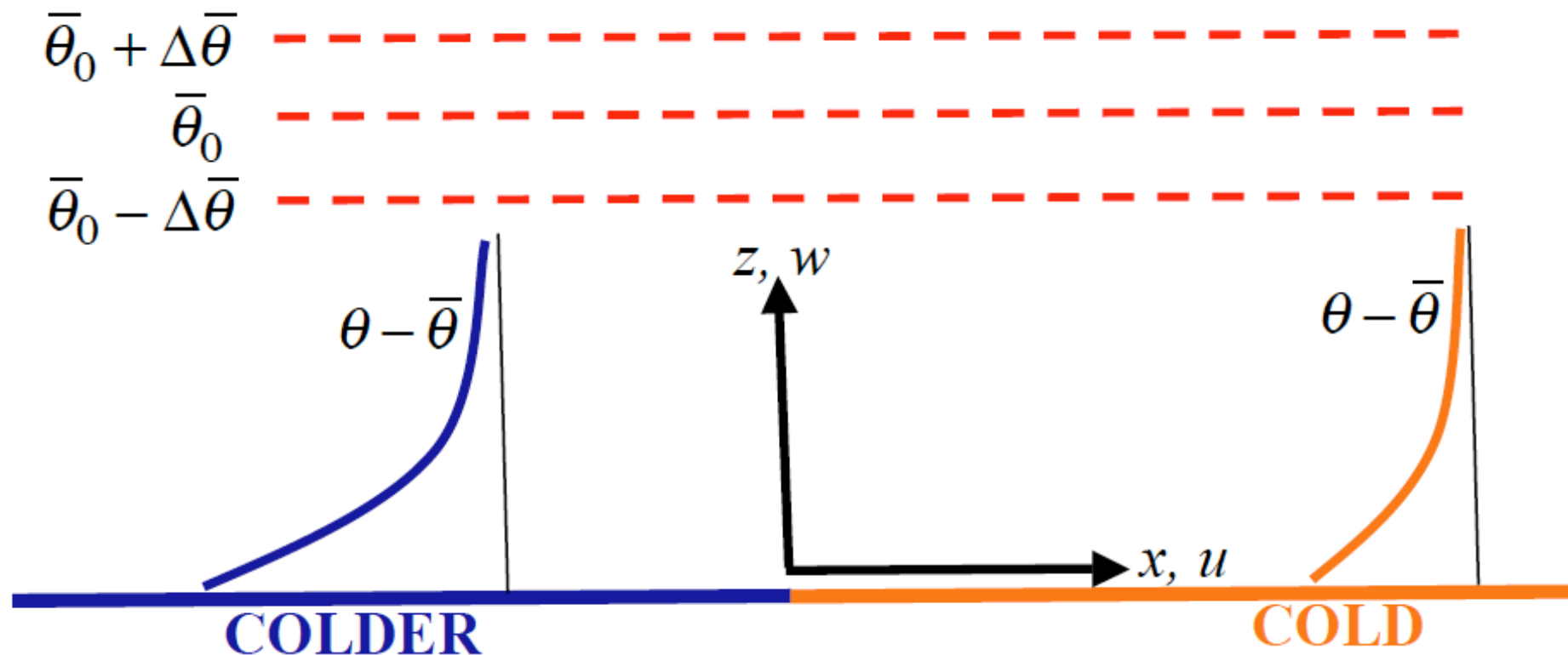
Horizontal natural convection is characterized by a shallow boundary layer flow.

Problem description

Fluid at rest fills the half space above an infinite horizontal plate at $z=0$.

At $t = 0$ a surface cooling is imposed: a steady temperature or heat flux that varies as a piecewise constant function of x with step change at $x = 0$.

Diffusion of heat from surface creates a lateral pressure gradient force that drives a shallow boundary-layer-like flow.



Boundary-layer equations (Boussinesq form)

Lateral (x) equation of motion:
$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -\frac{\partial \pi}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2}, \quad (1)$$

Hydrostatic equation:
$$0 = -\frac{\partial \pi}{\partial z} + b, \quad (2)$$

Thermal energy equation:
$$\frac{\partial b}{\partial t} + u \frac{\partial b}{\partial x} + w \frac{\partial b}{\partial z} = -\gamma w + \kappa \frac{\partial^2 b}{\partial z^2}, \quad (3)$$

Incompressibility condition:
$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \quad (4)$$

u and w : lateral (x) and vertical (z) velocity components

$\pi \equiv (p - \bar{p})/\rho_r$: normalized deviation of pressure p from environmental p

$b \equiv g(\theta - \bar{\theta})/\theta_r$: buoyancy, with θ = temperature or potential temperature

$\bar{\theta}(z)$ and $\bar{p}(z)$: environmental profiles of θ and p

$\gamma \equiv (g/\theta_r)d\bar{\theta}/dz$: stratification parameter (constant)

ν and κ : viscosity and thermal diffusivity (constant)

Vorticity equation

Taking $\partial/\partial z$ (1) – $\partial/\partial x$ (2) and using (4) yields the vorticity equation:

$$\left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + w \frac{\partial}{\partial z} \right) \frac{\partial u}{\partial z} = - \frac{\partial b}{\partial x} + v \frac{\partial^3 u}{\partial z^3}, \quad (5)$$

where $\partial u/\partial z$ is the y -component vorticity (boundary-layer approximated) and $-\partial b/\partial x$ is the baroclinic generation term.

To obtain a surface condition for vorticity, integrate (2) from $z = 0$ to ∞ :

$$\pi(x, 0) = - \int_0^{\infty} b(x, z) dz. \quad (6)$$

Then use (6) to evaluate (1) at the surface, obtaining

$$v \frac{\partial^2 u}{\partial z^2}(x, 0) = - \frac{\partial}{\partial x} \int_0^{\infty} b(x, z) dz. \quad (7)$$

Nondimensionalization

Buoyancy scale b_s based on surface buoyancy or buoyancy flux:

$$b_s \equiv \max_{x \in (-\infty, \infty)} |b(x, 0)|, \quad \text{or} \quad b_s \equiv \max_{x \in (-\infty, \infty)} \kappa^{1/2} [db/dz(x, 0)]^{3/4}. \quad (8)$$

Introduce ψ [$u = \partial\psi/\partial z$, $w = -\partial\psi/\partial x$] and the nondimensional variables

$$(X, Z) \equiv \frac{b_s^{1/3}}{\nu^{2/3}} (x, z), \quad T \equiv \frac{b_s^{2/3} t}{\nu^{1/3}}, \quad B \equiv \frac{b}{b_s}, \quad \Psi \equiv \frac{\psi}{\nu}, \quad \Gamma \equiv \frac{\gamma \nu^{2/3}}{b_s^{4/3}}, \quad \text{Pr} \equiv \frac{\nu}{\kappa}. \quad (9)$$

Equations (3) and (5) become the partial differential equations (PDEs)

$$\left(\frac{\partial}{\partial T} + \frac{\partial \Psi}{\partial Z} \frac{\partial}{\partial X} - \frac{\partial \Psi}{\partial X} \frac{\partial}{\partial Z} \right) B = \Gamma \frac{\partial \Psi}{\partial X} + \frac{1}{\text{Pr}} \frac{\partial^2 B}{\partial Z^2}, \quad (10)$$

$$\left(\frac{\partial}{\partial T} + \frac{\partial \Psi}{\partial Z} \frac{\partial}{\partial X} - \frac{\partial \Psi}{\partial X} \frac{\partial}{\partial Z} \right) \frac{\partial^2 \Psi}{\partial Z^2} = -\frac{\partial B}{\partial X} + \frac{\partial^4 \Psi}{\partial Z^4}. \quad (11)$$

3 governing parameters: Γ , Pr and ε (step change in surface forcing)

Group analysis of unstratified case

Consider the one-parameter (μ) family of stretching transformations

$$T' = \mu T, \quad X' = \mu^m X, \quad Z' = \mu^q Z, \quad \Psi' = \mu^r \Psi, \quad B' = \mu^s B, \quad (12)$$

The PDEs (10) and (11) (with $\Gamma = 0$) are invariant to (12) provided the exponents in (12) satisfy $q = 1/2$, $m = r + 1/2$, $s = 2r - 3/2$, $r = \text{arbitrary}$.

Seek solutions $B = g(X, Z, T)$, $\Psi = f(X, Z, T)$ that themselves are invariant to (12). By definition, such solutions satisfy $B' [= \mu^s B = \mu^s g(X, Z, T)] = g(X', Z', T')$ and $\Psi' [= \mu^r \Psi = \mu^r f(X, Z, T)] = f(X', Z', T')$, that is

$$\mu^{2r-3/2} g(X, Z, T) = g(\mu^{r+1/2} X, \mu^{1/2} Z, \mu T), \quad (13)$$

$$\mu^r f(X, Z, T) = f(\mu^{r+1/2} X, \mu^{1/2} Z, \mu T). \quad (14)$$

Take $d/d\mu$ of (13) and (14), and investigate the results at $\mu = 1$.

Similarity models for unstratified fluid

The group analysis shows that r controls the thermal surface condition. Only two values of r yield surface buoyancy or buoyancy flux distributions that are both **steady and well behaved** (non-singular) with respect to X . Those values yield the following **two similarity models**.

I. Unstratified fluid. Step change in surface buoyancy.

$$\Psi = T^{3/4} F(\xi, \eta), \quad B = G(\xi, \eta), \quad (15)$$

$$\xi \equiv XT^{-5/4}, \quad \eta \equiv ZT^{-1/2}. \quad (16)$$

II. Unstratified fluid. Step change in surface buoyancy flux.

$$\Psi = T F(\xi, \eta), \quad B = T^{1/2} G(\xi, \eta), \quad (17)$$

$$\xi \equiv XT^{-3/2}, \quad \eta \equiv ZT^{-1/2}. \quad (18)$$

Similarity model for stably stratified fluid

The group analysis for a stably stratified fluid ($\Gamma > 0$) yields specific values for all exponents. Fortuitously, these values are consistent with a surface thermal condition that is steady and well behaved with respect to X . These values yield the following similarity model.

III. Stratified fluid. Step change in surface buoyancy flux.

$$\Psi = T F(\xi, \eta), \quad B = T^{1/2} G(\xi, \eta), \quad (19)$$

$$\xi \equiv XT^{-3/2}, \quad \eta \equiv ZT^{-1/2}. \quad (20)$$

These scalings are identical to (17) and (18) that apply to a step change in the surface buoyancy flux in an unstratified fluid.

Propagation of solutions

The scalings constrain **all local maxima** in u , w , vorticity, convergence, and b gradient to **propagate with constant values of ξ and η** . We can thus easily infer the speeds and trajectories of those features.

Propagation of maxima in buoyancy forced flow I:

$$\frac{dX}{dT} = \frac{5}{4} \xi_c T^{1/4}, \quad \frac{dZ}{dT} = \frac{1}{2} \eta_c T^{-1/2}, \quad Z = \text{const } X^{2/5}. \quad (21)$$

Propagation of maxima in flux forced flows II and III:

$$\frac{dX}{dT} = \frac{3}{2} \xi_c T^{1/2}, \quad \frac{dZ}{dT} = \frac{1}{2} \eta_c T^{-1/2}, \quad Z = \text{const } X^{1/3}, \quad (22)$$

where ξ_c , η_c are constants defining a particular maximum.

Amplitude of solutions

As the local maxima propagate, their amplitudes change with time. The similarity scalings reveal these time dependencies.

Amplitude of select maxima in buoyancy forced flow I:

$$u \sim b_s^{1/2} \nu^{1/4} t^{1/4}, \quad w \sim \sqrt{\frac{\nu}{t}}, \quad \frac{\partial b}{\partial x} \sim \frac{b_s^{1/2}}{\nu^{1/4}} t^{-5/4}. \quad (23)$$

Amplitude of select maxima in flux forced flows II and III:

$$u \sim b_s^{2/3} \nu^{1/6} t^{1/2}, \quad w \sim \sqrt{\frac{\nu}{t}}, \quad \frac{\partial b}{\partial x} \sim \frac{b_s^{2/3}}{\nu^{1/3}} t^{-1}. \quad (24)$$

Time dependencies may also be easily deduced for peak convergence, peak vorticity, peak $\partial b/\partial z$ and many other derivative quantities.

PDEs for the similarity variables

Consider, for example, the stratified flow model **III**. Application of the similarity scalings $\Psi = T F(\xi, \eta)$ and $B = T^{1/2} G(\xi, \eta)$ in (10) and (11) yields

$$\left(\frac{1}{2} - \frac{3}{2} \xi \frac{\partial}{\partial \xi} - \frac{1}{2} \eta \frac{\partial}{\partial \eta} + \frac{\partial F}{\partial \eta} \frac{\partial}{\partial \xi} - \frac{\partial F}{\partial \xi} \frac{\partial}{\partial \eta} \right) G = \Gamma \frac{\partial F}{\partial \xi} + \frac{1}{\text{Pr}} \frac{\partial^2 G}{\partial \eta^2}, \quad (25)$$

$$\left(-\frac{3}{2} \xi \frac{\partial}{\partial \xi} - \frac{1}{2} \eta \frac{\partial}{\partial \eta} + \frac{\partial F}{\partial \eta} \frac{\partial}{\partial \xi} - \frac{\partial F}{\partial \xi} \frac{\partial}{\partial \eta} \right) \frac{\partial^2 F}{\partial \eta^2} = -\frac{\partial G}{\partial \xi} + \frac{\partial^4 F}{\partial \eta^4}. \quad (26)$$

Note that the dimension of the PDEs has been reduced from 3 (X, Z, T) to 2 (ξ, η).

Surface ($\eta = 0$) boundary conditions

No-slip:
$$\frac{\partial F}{\partial \eta}(\xi, 0) = 0 \quad (27)$$

Impermeability:
$$F(\xi, 0) = 0. \quad (28)$$

Vorticity constraint (7):
$$\frac{\partial^3 F}{\partial \eta^3}(\xi, 0) = -\frac{\partial}{\partial \xi} \int_0^\infty G(\xi, \eta) d\eta. \quad (29)$$

Step change in b :
$$G(\xi, 0) = \begin{cases} -1, & \xi < 0, \\ -\varepsilon, & \xi > 0. \end{cases} \quad (30)$$

or

Step change in b flux:
$$\frac{\partial G}{\partial \eta}(\xi, 0) = \begin{cases} \text{Pr}^{2/3}, & \xi < 0, \\ \varepsilon \text{Pr}^{2/3}, & \xi > 0. \end{cases} \quad (31)$$

Remote boundary conditions and initial conditions

The vorticity and buoyancy are considered to vanish far above the surface ($Z \rightarrow \infty$). Since $\eta \rightarrow \infty$ as $Z \rightarrow \infty$, these remote conditions become

$$\frac{\partial^2 F}{\partial \eta^2}, G \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty. \quad (32)$$

The vorticity and buoyancy are also considered to be zero initially. Since $\eta \rightarrow \infty$ as $T \rightarrow 0$, these initial conditions become

$$\frac{\partial^2 F}{\partial \eta^2}, G \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty, \quad (33)$$

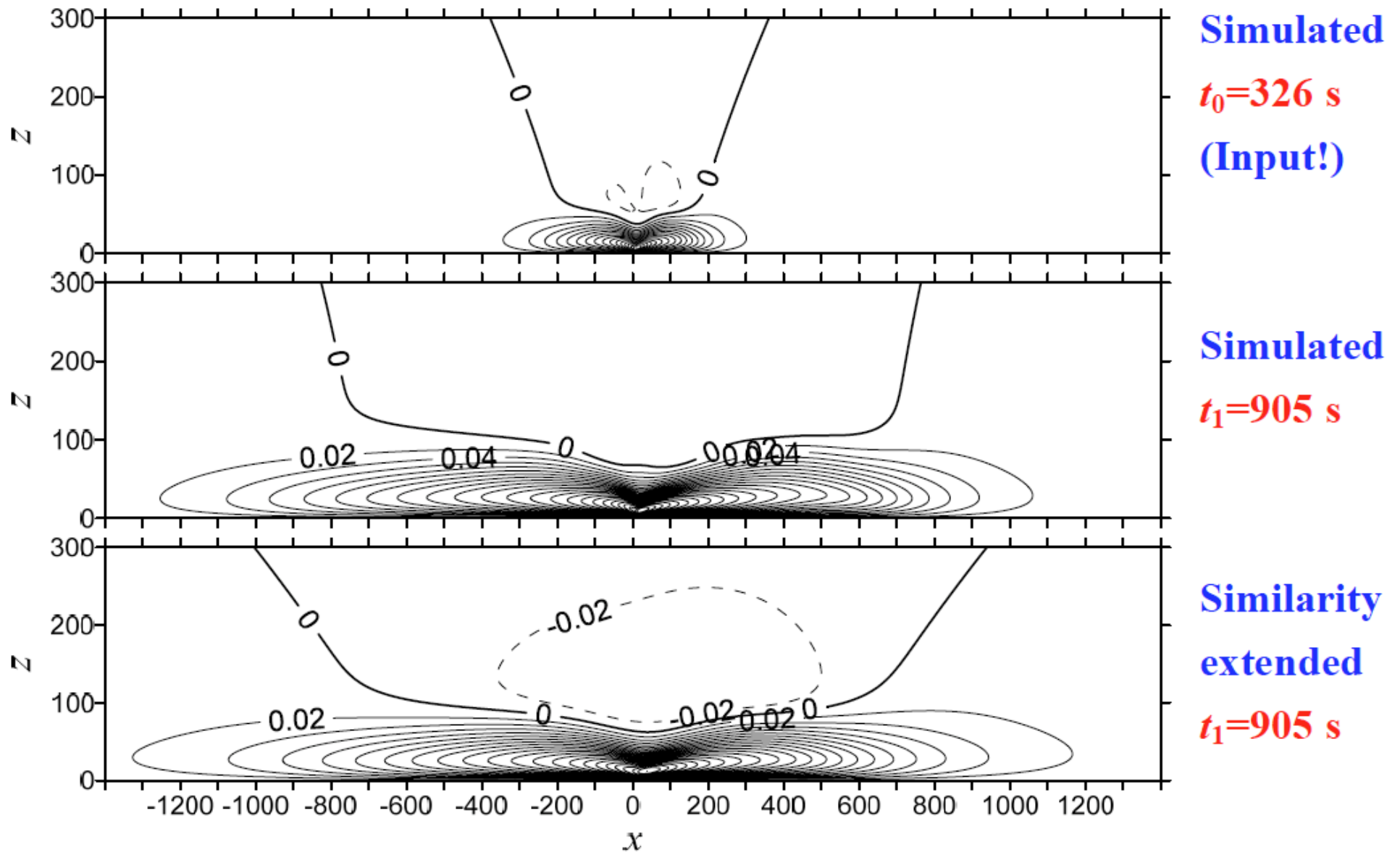
which is identical to (30). This conflation of initial and boundary conditions is typical of similarity models.

Validation tests

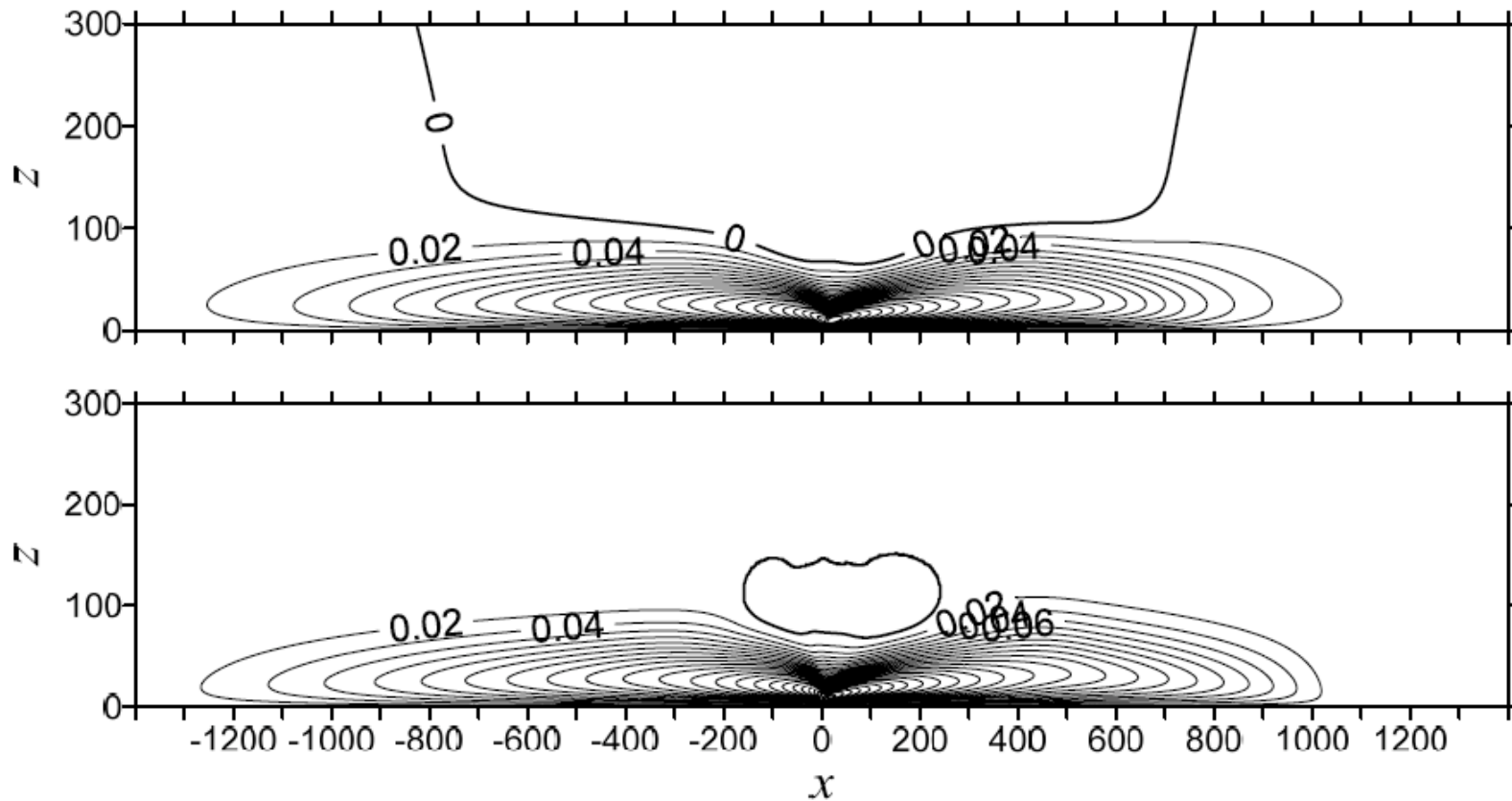
We have just begun to explore the solutions from the similarity models. Tests have thus far been restricted to $Pr = 1$ and $\varepsilon = 0.5$. Each test consists of the following procedures.

1. **Numerical simulation.** Generate a comparison (validation) data set by solving the Boussinesq equations of motion, thermal energy equation, and incompressibility condition numerically. **No boundary layer or hydrostatic approximations are made.**
2. **Similarity extension test.** Take a numerically simulated flow field from step 1 at an input time t_0 and extend it forward in time using the similarity scalings. Compare the extended and simulated fields at a second time t_1 .
3. **Similarity prediction test.** Solve the similarity PDEs iteratively. Compare the similarity predicted fields with the corresponding numerically simulated fields.

Buoyancy forcing, unstratified fluid: horizontal velocity u



Buoyancy forcing, unstratified fluid: prediction of u



Simulated

$t_1=905$ s

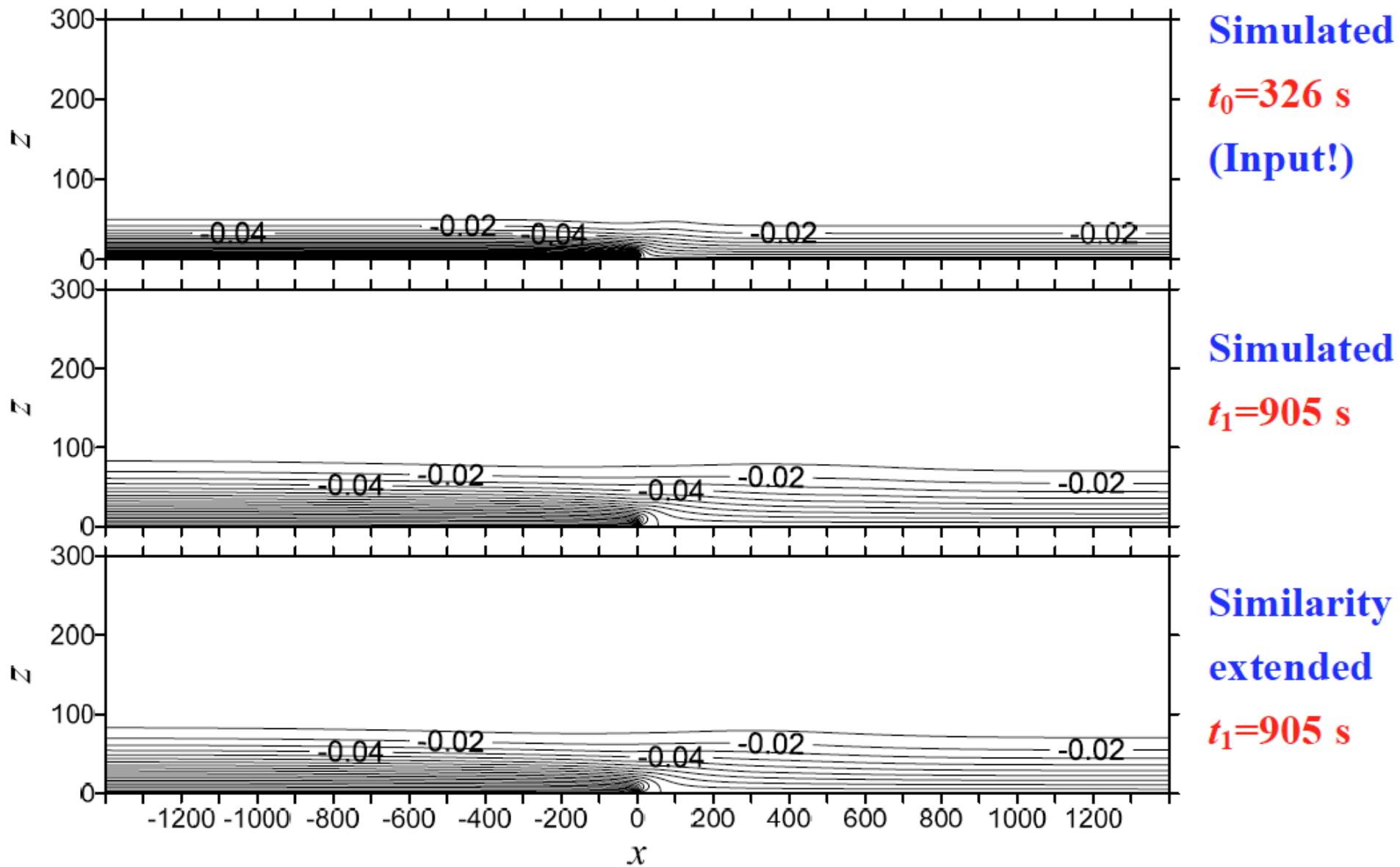
**Similarity
predicted**

$t_1=905$ s

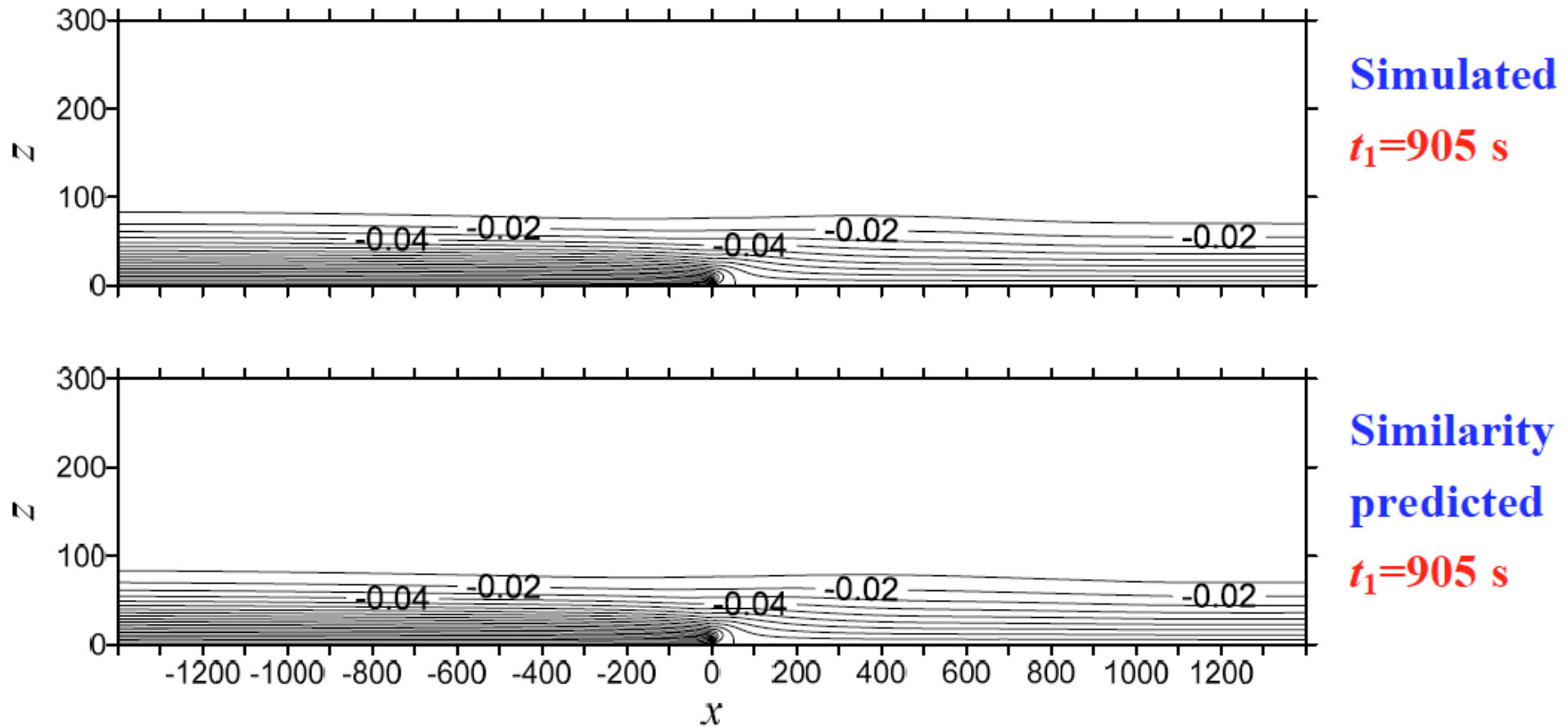
u_{\min} **Simulated:** **-0.020 m s^{-1}** ; **Predicted:** **-0.013 m s^{-1}** .

u_{\max} **Simulated:** **0.48 m s^{-1}** ; **Predicted:** **0.49 m s^{-1}** .

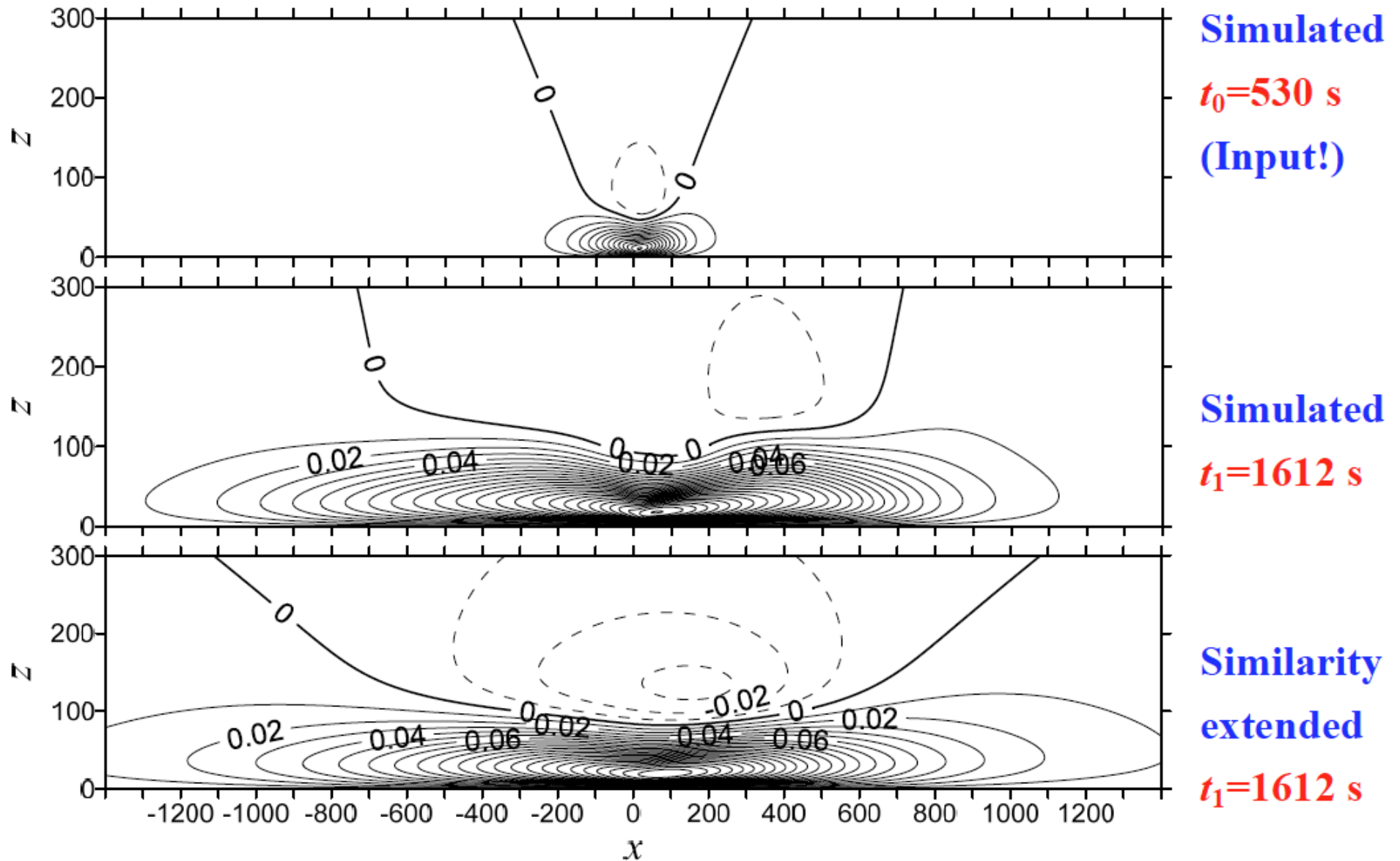
Buoyancy forcing, unstratified fluid: buoyancy b



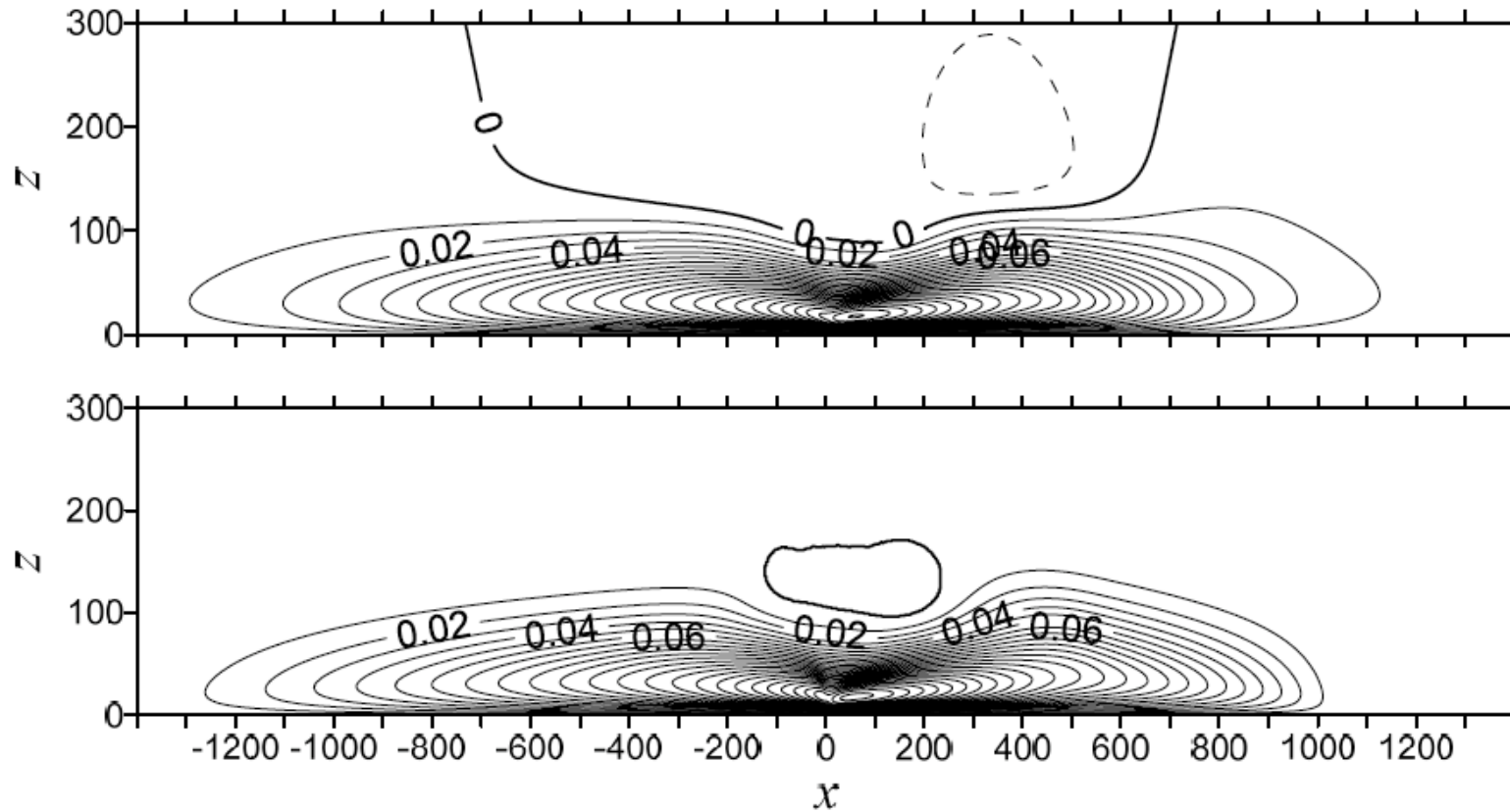
Buoyancy forcing, unstratified fluid: prediction of b



Flux forcing, unstratified fluid: horizontal velocity u



Flux forcing, unstratified fluid: prediction of u



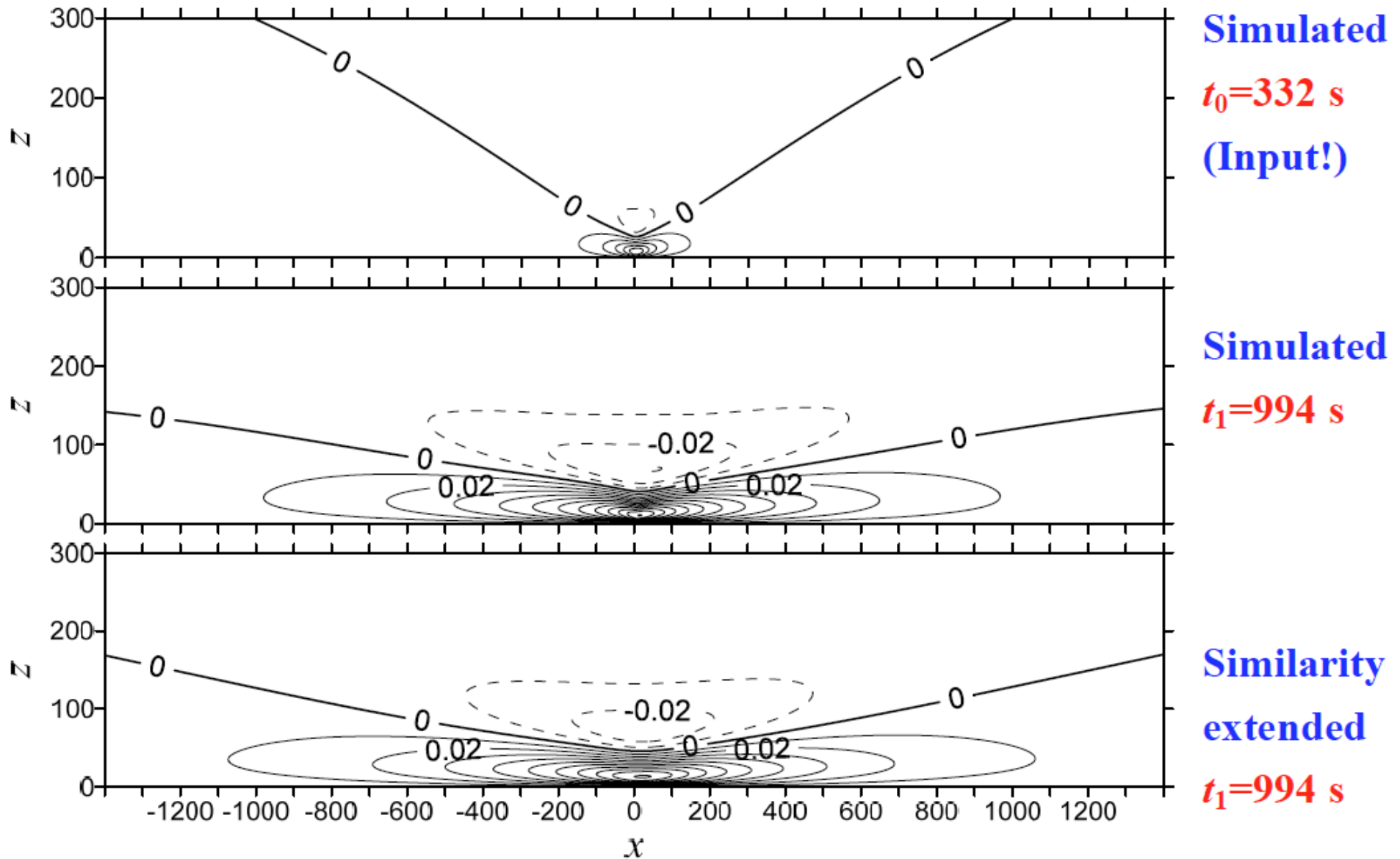
Simulated
 $t_1=1612$ s

Similarity
predicted
 $t_1=1612$ s

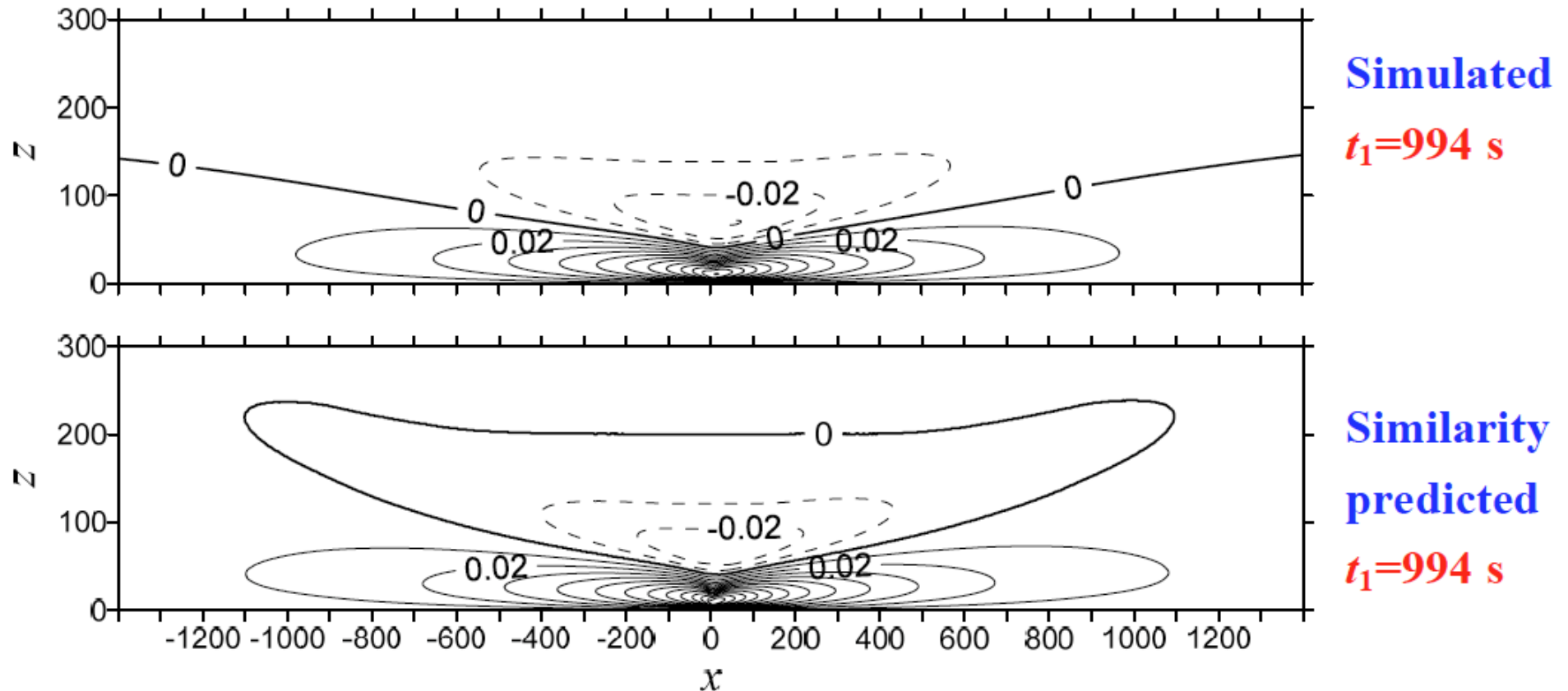
u_{\min} **Simulated: -0.015 m s^{-1} ; Predicted: -0.005 m s^{-1} .**

u_{\max} **Simulated: 0.25 m s^{-1} ; Predicted: 0.25 m s^{-1} .**

Flux forcing, stratified fluid: horizontal velocity u



Flux forcing, stratified fluid: prediction of u



u_{\min} **Simulated:** -0.030 m s^{-1} ; **Predicted:** -0.028 m s^{-1} .

u_{\max} **Simulated:** 0.12 m s^{-1} ; **Predicted:** 0.13 m s^{-1} .